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REVERSED RESIDUALS IN AUTOREGRESSIVE TIME SERIES ANALYSIS

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Both linear and nonlinear time series can have directional features, features which indicate that the series do not maintain identical statistical properties when the direction on the time scale is reversed. The main purpose of the present paper is to develop the analysis of these features and to indicate and illustrate how they can be used for the investigation and modelling of linear or nonlinear autoregressive statistical models. In particular, the aim of the paper is to introduce the idea of reversed residuals and to develop some of their properties. Particular pairs of reversed and ordinary residuals are shown to produce partial autocorrelation coefficients: quadratic types of partial autocorrelation coefficients are introduced to assess dependence associated with nonlinear models which nevertheless have linear autoregressive (Yule-Walker) correlation structures.

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REVERSED RESIDUALS IN AUTOREGRESSIVE TIME SERIES ANALYSIS

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1 Introduction

Both linear and nonlinear time series can have directional features, features which indicate that the series do not maintain identical statistical properties when the direction on the time scale is reversed. The main purpose of the present paper is to develop the analysis of these features and to indicate and illustrate how they can be used for the investigation and modelling of linear or nonlinear autoregressive statistical models. In particular, the primary aim of the paper is to introduce the idea of reversed residuals (mentioned in discussion to Green (1984) and Lawrance and Lewis (1985)), and to develop some of their properties. Reversed residuals in general are residuals which would have been obtained if time ran in the opposite direction and arise naturally when considering partial autocorrelations for time series; cross correlations of residuals and squared-reversed residuals allow extensions of model identification ideas given in Lawrance and Lewis (1986). Particular pairs of reversed and ordinary residuals are shown to produce partial autocorrelation coefficients: quadratic types of partial autocorrelation coefficients are introduced to assess dependence associated with nonlinear models which nevertheless have linear autoregressive (Yule-Walker) correlation structures.

A parallel theoretical study in this paper concerns the use of reversed residuals in the investigation of random coefficient autoregressive models (Andel, 1976, 1983; Vervaat, 1979; Nicholls and Quinn, 1982); the class includes the NEAR(1) models of Lawrance and Lewis (1981) with exponential marginals, the BGAR(1) models of Lewis et al (1989) with gamma marginals and the PBAR models of McKenzie (1985) with beta marginals. A result for all such models is given concerning a cut-off property of quadratic cross-correlations of the ordinary and reversed residuals: a partial reversibility condition in terms of moments for these models is also stated.

The relevance of our concerns here with directionality is due to the fact that most nonlinear processes are directional, and amongst linear autoregressive processes, the

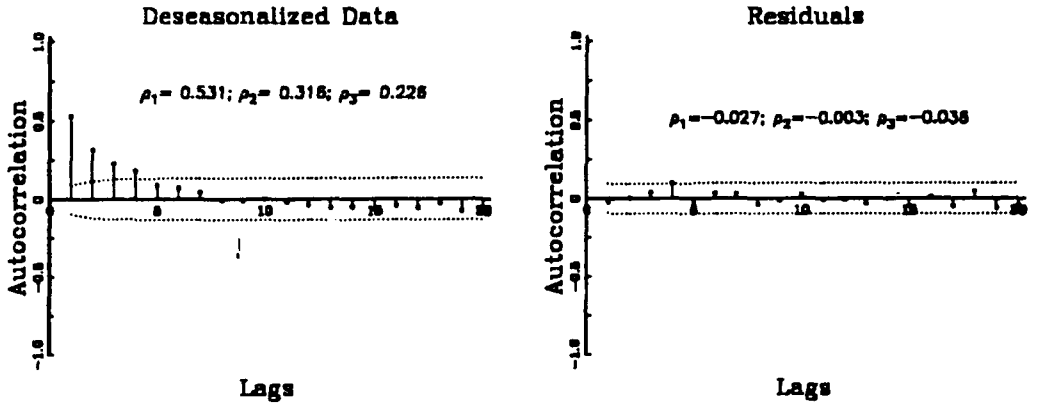

Figure 1: Autocorrelation functions of deseasonalized Stour river data and the residuals from a first order autoregressive fit to that data. The bands define approximate confidence intervals on the autocorrelations for lag greater than zero on the assumption that the true autcorrelation function is zero for lags greater than zero.

only non-directional or reversible ones are Gaussian. This latter fact forms part of a central result due to Weiss (1975) and indicates that use of reversed residuals is also of relevance to assessing the linearity of non Gaussian autoregressive processes.

The use of reversed residuals is illustrated on a series of deseasonalized monthly British riverflow data in which it is shown that there is some nonlinear first order autoregressive dependency. The data is monthly flow data of the River Stour, Stourport, Gloucestershire, England, from 1918 to 1945; there are 444 data values in all. Following common hydrological practice, it has been deseasonalized by standardizing each value by subtracting its own monthly mean and dividing by its own monthly standard deviation. This is effective in reducing the flows to a stationary series; the marginal distribution of the deseasonalized data is non Gaussian and very positively skewed. The autocorrelation is reasonably geometric as can be seen from the left hand panel in Figure 1, and the partial autocorrelation function (not shown) cuts-off at lag 2 as expected. Furthermore, the fit of a first order linear autoregressive model is satisfactory as judged by the autocorrelation function of its residuals, shown in the right hand panel of Figure 1. More formally, testing shows that the cumulated periodogram of the residuals is consistent with an hypothesis of uncorrelated residuals. This result is confirmed by modified Box-Pierce chi-square statistics at 12, 24, 36 and 48 lags.

Reversed autoregressive residuals are introduced in the next section. The purpose of this paper is to see what further can be said about autoregressive fits, and in particular about the first order fit to the Stour river data, by using third and fourth joint properties of both the residuals and reversed residuals. For this purpose cross correlations of residuals and squared reversed residuals, as well as other combinations of the residuals, are examined. In addition, quadratic types of partial correlation coefficients are introduced.

2 REVERSED RESIDUALS

Consider the time series $\{X_t\}$ which is stationary with mean zero; the p^{th} order autoregressive residual sequence for this series is defined as

$$R_t^{(p)} = X_t - \alpha_1 X_{t-1} - \ldots - \alpha_p X_{t-p}, \qquad t = 0, \pm 1, \pm 2, \ldots \tag{1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_p$ are chosen to minimize

$$E(X_t - \alpha_1 X_{t-1} - \alpha_2 X_{t-2} - \ldots - \alpha_p X_{t-p})^2. \tag{2}$$

If the true model for $\{X_t\}$ is autoregressive of order p, then $\alpha_1, \alpha_2, \ldots, \alpha_p$ are its coefficients and R_t^p is its innovation term; in general $\alpha_1, \alpha_2, \ldots, \alpha_p$ are given by p linear equations of the Yule–Walker type involving the autocorrelations of $\{X_t\}$. The reversed p^{th} order autoregressive residuals of $\{X_t\}$ are correspondingly defined, with the index t increasing, as

$$RR_t^{(p)} = X_t - \beta_1 X_{t+1} - \beta_2 X_{t+2} - \ldots - \beta_p X_{t+p} \qquad t = 0, \pm 1, \pm 2, \ldots \tag{3}$$

and $\beta_1, \beta_2, \ldots, \beta_p$ minimizing

$$E(X_t - \beta_1 X_{t+1} - \beta_2 X_{t+2} - \ldots - \beta_p X_{t+p})^2. \tag{4}$$

The β's thus satisfy an identical set of linear equations to those of $\alpha_1, \alpha_2, \ldots, \alpha_p$. Calculation of the residuals and reversed residuals will be based on ordinary least squares regression rather than the nearly equivalent, Gaussian theory, first order autoregressive likelihood.

Suppose the available data series, after correction for its mean $\overline{x}$ is given by $(x_1, x_2, \ldots, x_n)$. It is convenient when dealing with p^{th} order residuals to form the display

$$\begin{array}{cccccc}
x_1 & x_2 & x_3 & \ldots & x_p & x_{p+1} \\
x_2 & x_3 & x_4 & \ldots & x_{p+1} & x_{p+2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n-p-1} & x_{n-p} & x_{n-p+1} & \ldots & x_{n-2} & x_{n-1} \\
x_{n-p} & x_{n-p+1} & x_{n-p+2} & \ldots & x_{n-1} & x_n
\end{array}$$

The p^{th} order reversed residuals are then formed as the residuals from regressing the first column on the p subsequent columns; similarly, p^{th} order ordinary residuals come from regressing the $(p+1)^{th}$ column on the p previous columns. In this way, the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_p$ in (1) and $\beta_1, \beta_2, \ldots, \beta_p$ in (3) are implicitly estimated, but never needed explicitly: they will not be estimated as equal, which theoretically they are, in order to make most use of the available data. An alternative procedure would be to use one or a combination of the two estimated sets for both residuals and reversed residuals ; there are implications of these comments to the calculation of partial correlation coefficients, to be given in Section 5.

An alternative procedure, mentioned initially could be to assume a standard p^{th} order linear Gaussian autoregressive model for the $\{X_t\}$; however, there is no real

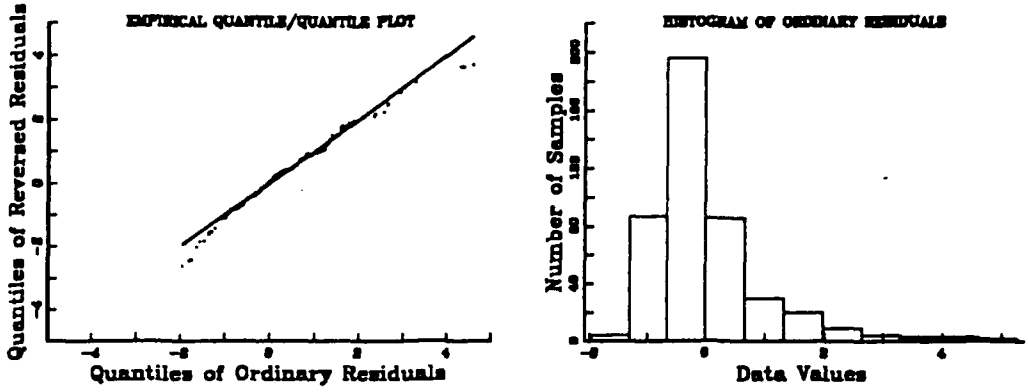

Figure 2: Histogram for the ordinary residuals, and empirical quantile-quantile plot for the ordinary and reversed residuals for the Stour data.

justification for this, except that the maximum likelihood estimates of the coefficients would correspond very closely to the regression estimates of $\alpha_1, \alpha_2, \ldots, \alpha_p$. Fitting to the reversed series would similarly give estimates of the parameters $\beta_1, \beta_2, \ldots, \beta_p$.

The structure of reversed residuals is rather complicated even in the situation when the order considered is that of the true linear autoregressive model; they do not have the same distribution as that of the ordinary residuals and they are not serially independent. It is straightforward to see, however, as in Lawrance and Lewis (1986) for the ordinary residuals, that the reversed residuals are uncorrelated under the linearly correct order of autoregression, that is Yule-Walker autoregression; moreover, they have the usual cut off property at the correct order of autoregressions as associated with the ordinary residuals of autoregressive models. The cross correlation function of the ordinary and reversed residuals is also zero at negative lags less than the degree of Yule–Walker autoregression. These features are all evident in the corresponding plots of the Stour data; a histogram of the ordinary residuals is given in the right hand panel in Fig 2. The distributions of both the ordinary and reversed residuals for these data are compared in the empirical quantile-quantile plot shown in the left hand panel in Figure 2. The distributions differ significantly in the tails. Moreover, since the average of each data set is approximately zero, it is seen that both data sets are positively skewed and thus non-normal.

To illustrate the structure of reversed residuals consider the linear AR(1) model given by

$$X_t = \rho X_{t-1} + \epsilon_t. \tag{5}$$

Then

$$\begin{aligned} RR_t^{(1)} \equiv X_t - \rho X_{t+1} &= (1-\rho^2)X_t - \rho\epsilon_{t+1} \\ &= -\rho\epsilon_{t+1} + (1-\rho^2)\left\{\epsilon_t + \sum_{r=1}^{\infty}\rho^r\epsilon_{t-r}\right\}. \end{aligned} \tag{6}$$

It is seen that for this linear AR(1) model $RR_t^{(1)}$ is dependent on $\epsilon_{t+1}, \epsilon_t, \epsilon_{t-1}, \ldots$ but that it will be independent of $\epsilon_{t+2}, \epsilon_{t+3}, \ldots$, that is, independent of the ordinary

residuals $R^{(1)}_{t+2}, R^{(1)}_{t+3}, \ldots$. However, the $\left\{RR^{(1)}_t\right\}$ sequence itself is dependent at all lags, unless the $\{\epsilon_t\}$ are Gaussian.

The main motivation for reversed residuals in general is that they capture directionality in a meaningful manner and allow model validation or criticism to be extended both beyond standard linear methods and differently from higher order methods involving ordinary residuals. The relevant class of models will naturally include a Yule–Walker autoregressive aspect, but will in general be nonlinear. The correct order of this linear autoregression is assumed, as determined by conventional means. The autoregressive aspect then leads to uncorrelated reversed residuals and the suggestion is to obtain measures of their higher order dependency which can be compared with corresponding estimated quantities. We will suggest some "quadratic" correlations involving the squaring of residuals and reversed residuals. An assessment of reversibility can also be based on the quadratic correlation function.

3 USEFUL QUANTITIES IN TERMS OF RESIDUALS AND REVERSED RESIDUALS

In a previous paper (Lawrance and Lewis, 1986), use was made of higher order correlations of the ordinary residuals as a means to help identify nonlinear autoregressive processes which, nevertheless, had Yule–Walker linear autoregressionss. In particular attention was directed at $Corr\left\{R^{(p)}_t, \left(R^{(p)}_{t+r}\right)^2\right\}$; for a p^{th} order standard linear autoregressive process these correlations are zero, and for a nonlinear process with p^{th} order Yule–Walker autoregressions the residuals $\left\{R^{(p)}_t\right\}$ are uncorrelated but dependent. Thus the higher order correlations, which will be called quadratic correlations when they involve squaring, —such as that just cited—give an assessment of the uncorrelated dependence; Granger and Anderson (1978) first considered the autocorrelations of squared residuals. As a data analysis tool, these correlations would be estimated for a range of lags for given p, the value of p having already been determined by a Yule–Walker linear analysis. The further analysis is then concerned with diagnosing nonlinear features in the autoregressions. Model based calculations of these quadratic correlations allow an assessment of whether the data is in reasonable agreement with the (non-linear) autoregressive model.

With the introduction of reversed residuals there are three further possible pairs of quadratic correlations which could be examined, that is correlations of

$$\left(R^{(p)}_t, RR^{(p)^2}_{t+r}\right), \left(R^{(p)^2}_t, RR^{(p)}_{t+r}\right), \left(RR^{(p)}_t, RR^{(p)^2}_{t+r}\right), \qquad r = \pm 1, \pm 2, \ldots \qquad (7)$$

It still seems to be a matter of experience which of these is most useful in model validation; when considering them for a range of positive and negative values of r, they are all basically functions of third moments of the form $E\left(X^3_t\right), E\left(X^2_t X_{t+r}\right)$ and $E\left(X_t X_{t+r} X_{t+s}\right)$. One possible guide to their use could be any special properties which might be exploited in relation to the type of model of interest; properties of

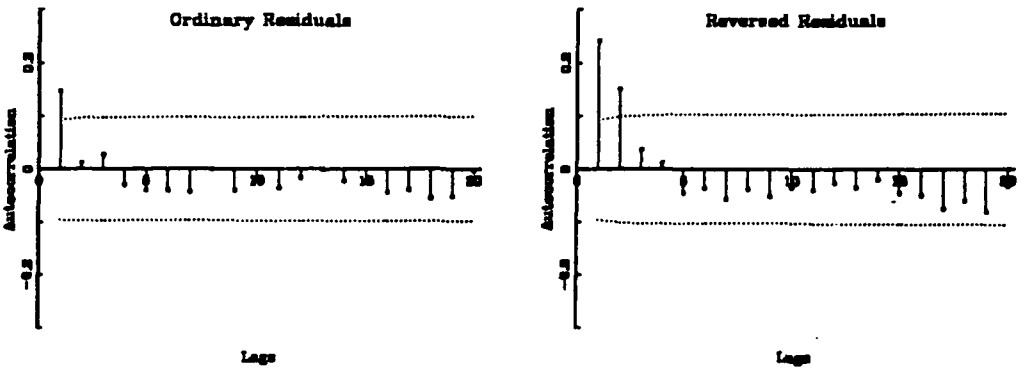

Figure 3: Autocorrelation function of squared ordinary and reversed residuals for the Stour data. The bands define approximate confidence intervals on the autocorrelations for lag greater than zero on the assumption that the true autcorrelation function is zero for lags greater than zero.

these correlations for linear and random coefficient first order autoregressive models will be considered in Section 5.

For the Stour data, the autocorrelation function of the squared ordinary and reversed residuals are given in Figure 3. The autocorrelation at lag 1 of the ordinary squared residuals is evidence here of the non-linearity in the series. The nonlinearity is also picked up in the autocorrelation function of the squared reversed residuals. A matrix of the third order cross-correlation functions, three of which are given in (7) is given in Figure 4 and will be interpreted against the theoretical results for linear and random coefficient models in Section 6.

Another use of these quantities will be in connection with detecting directionality. For instance the marginal distributions of $R_t^{(p)}$ and $RR_t^{(p)}$ will be equal for reversible processes and the scatters of $R_t^{(p)}$ and $RR_t^{(p)}$ should be symmetric about the 45^0 line. If reversibility is restricted to relevant moment conditions then the correlation function of the first two pairs in (7) should be mirror images of each other; similarly, the last pair of (7) should equal the corresponding quantity in terms of ordinary residuals.

It is seen from Figure 4 that for the Stour data there is no evidence for reversibility. In particular, the plot in the upper right quadrant, even allowing for sampling variability, is different from that in the lower left quadrant. So are the plot in the upper left quadrant and the plot in the lower right quadrant, as predicted if there is non-linearity in the data.

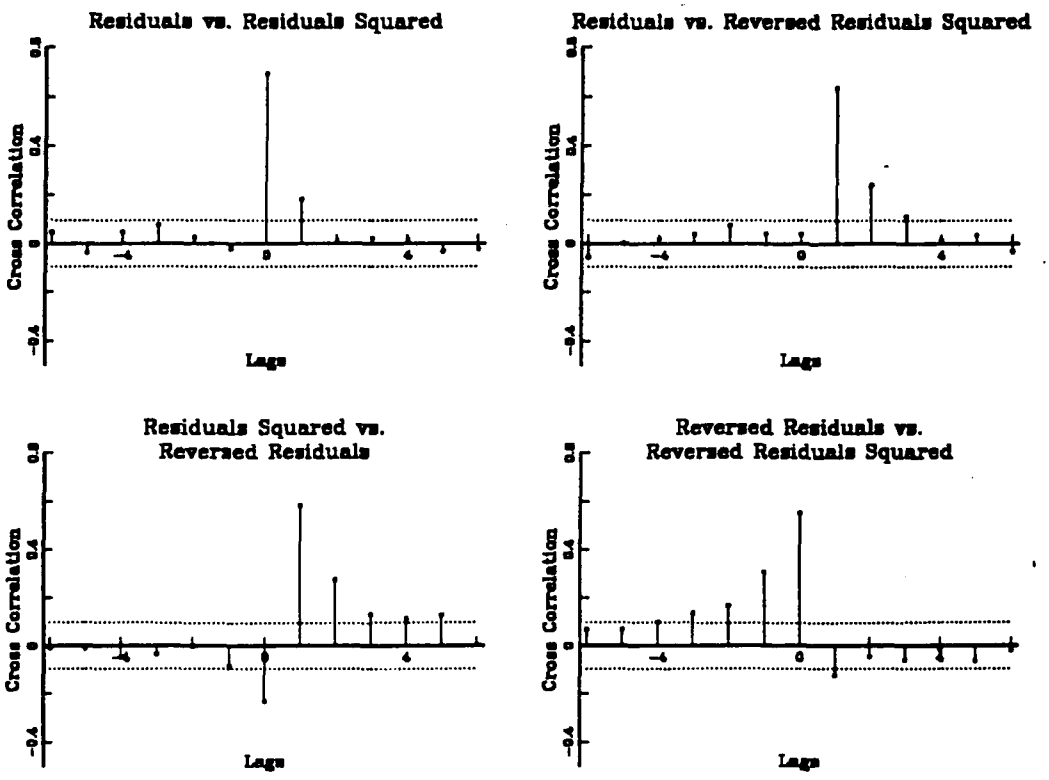

Figure 4: Crosscorrelation functions for the third order dependency of ordinary and reversed residuals for the Stour data. Bands are individual 95% confidence intervals for the crosscorrelation estimates under the hypothesis of independence between the two series.

4 QUADRATIC TYPES OF PARTIAL CORRELATION COEFFICIENTS

The introduction of reversed residuals allows the familiar partial autoregression function of time series analysis to be seen to some advantage. A discussion of the usual indirect definition of partial autocorrelations involving autoregressive models, and advocacy of the direct definition, is given in Lawrance (1979); it is pointed out there that a partial autocorrelation is the ordinary correlation between two random varaiables after their linear dependence on the partialed out random variables has been subtracted out. Thus taking the direct approach here, consider X_t and X_{t+p}; the p^{th} partial autoregressions can be defined as the correlation between X_t and X_{t+p} after each has been adjusted, in the least squares sense, for its linear dependence on the intervening $X_{t+1}, X_{t+2}, \ldots, X_{t+p-1}$. The adjusted X_t and X_{t+p} are then simply seen to be the reversed residual $RR_t^{(p-1)}$ and the ordinary residuals $R_{t+p}^{(p-1)}$, respectively. In the partial correlation of X_t and X_{t+p} notice that $p(p \geq 2)$ is the lag and $p-1$ is the order of the residuals used; the lag 1 partial autoregressions is,

by convention, the ordinary lag 1 autoregressions. Partial autoregressions will often be calculated or estimated for $p = 2,3,\ldots$ and used in determination of p, rather than with p held constant as we have previously suggested in higher order model validation.

The earlier introduction of quadratic cross correlations suggest the use of two quadratic partial autoregressions of the form

$$Corr\left\{RR_t^{(p-1)}, \left(R_{t+p}^{(p-1)}\right)^2\right\}, Corr\left\{\left(RR_t^{(p-1)}\right)^2, R_{t+p}^{(p-1)}\right\}, \quad p = 2,3,\ldots. \qquad (8)$$

As with partial autoregressions for standard linear autoregressive models, both these quadratic partial autoregressions will have cut-offs at the true autoregressive order; they will thus assess the linearity of the autoregressive structure, maybe indicating that a non-linear autoregressive model is required. For the Stour data, calculation of the quadratic partial correlations indicate no autoregressive dependency beyond lag one. At lag one ($p = 2$ in (8)), the first cross correlation has a value of -0.121. For $p = 2$, the cross correlations of (8) are effectively zero. The cut-off property does not depend on the direction of autoregression.

The computation of all the partial autocorrelation functions follows directly from the least squares calculations described in Section 3 to obtain the ordinary and reversed residuals. This method of calculating the standard partial autoregressions will usually produce slightly different results from the traditional method of taking the last estimated coefficients in a linear Gaussian autoregressive model. It is equivalent to estimating only the α-set of coefficients of the ordinary residuals and using these also to construct the reversed residuals.

5 REVERSED RESIDUALS FOR RANDOM COEFFICIENT AUTOREGRESSIVE MODELS

A simple nonlinear generalization of standard linear autoregressive models is to allow the coefficients to be random variables; the general first order form of these models is then

$$X_t = A_t X_{t-1} + B_t, \qquad t = 0, \pm 1, \pm 2, \ldots \qquad (9)$$

where $\{X_t\}$ has mean μ and $\{A_t, B_t\}$ are independent pairs of possibly dependent random variables. A number of particular models fit into this class, for instance, the exponential models of Lawrance and Lewis (1981), the gamma models of Lewis et al (1989) and the beta models of McKenzie (1985). Currently these models are well developed for simulation use but are less developed in their statistical aspects. Lawrance and Lewis (1985, 1986) considered residuals analysis based on ordinary residuals for this type of model, and here attention is directed at reversed residuals aspects.

Denoting the first two moments of $\{A_t\}$ by a and a_2, the first order ($p = 1$) residuals are given by

$$R_t = (X_t - \mu) - a(X_{t-1} - \mu), \quad RR_t = (X_t - \mu) - a(X_{t+1} - \mu). \qquad (10)$$

Some simple calculations show that the cross correlations function of these residuals is given for the first-order autoregressive random coefficient model by

$$Corr\,(R_t, RR_{t-r}) = \begin{cases} (1-a^2)a^{|r|} & r \leq 0 \\ -a & r = 1 \\ 0 & r \geq 2 \end{cases}. \tag{11}$$

The form of this result has been anticipated in Section 3, but of more interest are the quadratic cross correlation functions (7). For (R_t^2, RR_{t-k}), the following results can be obtained; the computations are straightforward but very complex and tedious:

$$\begin{aligned} &Cov\left(R_t^2, RR_{t-r}\right) \\ &= (a_2 - a^2)(a_2 - a)\left\{\mu_3 + k/(a_2 - a)\right\} a_2^{r-2}, && r \geq 2 \quad (12) \\ &= \left[\left(1 - a + a^3 - a^5\right)\mu_3 - 2a\left(1 - a^2\right)(a_2\mu_3 + k)\right] a^{|r|}, && r \leq -1 \quad (13) \\ &= \left(1 + a^2 - a^3 - a^5\right)\mu_3 - 2a\left(1 - a^2\right)(a_2\mu_3 + k), && r = 0 \quad (14) \\ &= \left(-a - a^2 - a^4\right)\mu_3 + \left(1 + 2a^2\right)(a_2\mu_3 + k), && r = 1 \quad (15) \end{aligned}$$

The further notation here is that $\mu_3 = E(X_t - \mu)^3, k = 2\sigma^2\{\mu \times var(A_t) + Cov(A_t, B_t)\}$ and $\sigma^2 = var(X_t)$. This result is of interest because its geometric parameter for $r \geq 2$ is a_2 and so the corresponding estimated correlation function would give a possible way to estimate the variance of the random coefficients A_t; further, as will be seen in Section 6, $\mu_3 + k/(a_2 - a)$ is a measure of the directionality of the model. Another implication of (12) is that the cross correlations, for $r \geq 2$, are zero for linear processes. This is because $var(A_t) = a_2 - a^2 = 0$ and therefor $a_2 = a^2$; this is not so for $r < 2$, as shown by by (13)- (15). For the Stour data, Figure 4 (lower left quadrant) shows that the cross correlations corresponding to (12) are all positive, and although small, provide further evidence against linearity of this data.

For the p^{th} order random coefficient autoregressive model,

$$X_t = A_t^{(1)}X_{t-1} + A_t^{(1)}X_{t-2} + \ldots + A_t^{(p)}X_{t-p} + B_t, \tag{16}$$

generalizing (9), there is a parallel to the residuals theorem in Lawrance and Lewis (1986). There is the following

Theorem: With the random coefficient model (16)

$$Corr\left[R_t^{(p)}, \left\{RR_{t-r}^{(p)}\right\}^2\right] = 0 \quad \text{for} \quad r \geq p+1. \tag{17}$$

Proof: This depends on the independence of the vector of coefficients $\left\{A_t^{(1)}, A_t^{(2)}, \ldots, A_t^{(p)}, B_t\right\}$ on previous X_t's. We have

$$\begin{aligned} R_t &= X_t - \mu - \alpha_1(X_t - \mu) - \ldots - \alpha_p(X_{t-p} - \mu) && (18) \\ &= (A_t^{(1)} - \alpha_1)X_{t-1} + \left(A_t^{(2)} - \alpha_2\right)X_{t-2} + \ldots + \left(A_t^{(p)} - \alpha_p\right)X_{t-p} \\ &\quad + B_t - (1 - \alpha_1 - \alpha_2 \ldots - \alpha_p)\mu && (19) \end{aligned}$$

where $E\left(A_t^{(i)}\right) = \alpha_i$. Multiplying (16) by $\left\{RR_{t-r}^{(p)}\right\}^2$ and taking expectations gives

$$\sum_{j=1}^{p} E\left[(A_t^{(j)} - \alpha_j)X_{t-j}\left\{RR_{t-r}^{(p)}\right\}^2\right] + E\left[\left\{B_t - \left(1 - \sum_{i=1}^{p}\alpha_i\right)\mu\right\}\left\{RR_{t-r}^{(p)}\right\}^2\right]. \quad (20)$$

Now $RR_{t-r}^{(p)}$ involves $X_{t-r}, X_{t-r+1}, \ldots, X_{t-(r-p)}$; when $r \geq p+1$ this sequence is independent of $A_t^{(j)}$, as is X_{t-j}. Since $A_t^{(j)} - \alpha_j$ has zero expectation, so does the first summation in (20), and the second is also zero by very similar arguments. The random coefficients can still be dependent within each t for the argument to remain valid. The theorem is proved since (20) is the covariance corresponding to (17). □

6 REVERSIBILITY FOR FIRST ORDER RANDOM COEFFICIENT AUTOREGRESSIVE MODELS

The first order random coefficient autoregressive model (9) will be fully reversible when the joint distribution (X_t, X_{t-1}) is symmetric; this follows from results of McKenzie (1985) for first order Markov models. However, although this joint distribution was obtained explicitly for the gamma model of Lewis et al (1989), thus establishing the reversibility of this process, the joint distribution cannot be obtained generally in an explicit enough form to yield a tractable condition. We thus revert to the simplest moment form of partial reversibility, defined by the requirement that

$$E\left\{(X_t - \mu)(X_{t-r} - \mu)^2\right\} = E\left\{(X_t - \mu)^2(X_{t-r} - \mu)\right\}, \quad r = 0, \pm 1, \pm 2, \ldots \quad (21)$$

For the first order random coefficients model (9) expressions for these joint moments are given in Lawrance and Lewis (1986, equations (4.5) for $r \geq 1$ and (4.13) for $r \leq -1$) as $a_2^r\mu_3$ and $a_2^r\mu_3 + k(a_2^r - a^r)/(a_2 - a)$, respectively. The equality of these directional moments, for all r, yields the condition

$$\mu_3 + k/(a_2 - a) = 0. \quad (22)$$

This condition simplifies a little when A_t and B_t are independent and can then be written as

$$\mu_3 = 2\mu\sigma^2 var(A_t)/\{a(1-a) - var(A_t)\}. \quad (23)$$

Note in particular that the skewness of X_t is non-zero unless $var(A_t) = 0$, as it is in the case of the linear model; the skewness can otherwise be zero when A_t and B_t are correlated.

It can be seen from (12) that $Corr\,(R_t^2, RR_{t-r})$ for $(r \geq 2)$ is zero when the process satisfies the reversibility condition (22); this could be a useful result in model validation. In this respect also, consideration of $Corr\,(R_t^2 R_{t-r})$ as derived in Lawrance and Lewis (1986, equation (4.14)), is relevant. This can be cast in the new form, for $r \geq 1$, of

$$Cov(R_t^2, R_{t-r}) = \left(\frac{a_2 - a^2}{1 - aa_2}\right)\left\{\mu_3 + \frac{k}{a_2 - a}\right\} a_2^{r-1} - \frac{a(1-a)(1-a^2)}{a_2 - a}\, k\, a^{r-1}. \quad (24)$$

Thus, under the reversibility condition (22), the first term is zero and $Corr(R_t^2, R_{t-r})$ is geometrically decreasing in the parameter a. It is such special simplifications which justify the consideration of more than one of the mathematically equivalent cross correlation functions described in Section 4.

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